Characteristics of Minimal Effective Programming Systems*

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Abstract. The Rogers semilattice of effective programming systems (epses) is the collection of all effective numberings of the partial computable functions ordered such that $\theta \leq \psi$ whenever θ-programs can be algorithmically translated into ψ-programs. Herein, it is shown that an eps ψ is minimal in this ordering if and only if, for each translation function t into ψ, there exists a computably enumerable equivalence relation (ceer) R such that (i) R is a subrelation of ψ's program equivalence relation, and (ii) R equates each ψ-program to some program in the range of t. It is also shown that there exists a minimal eps for which no *single* such R does the work for all such t. In fact, there exists a minimal eps ψ such that, for each ceer R, either R contradicts ψ's program equivalence relation, or there exists a translation function t into ψ such that the range of t *fails* to intersect *infinitely many* of R's equivalence classes.

Keywords: computably enumerable equivalence relation, Friedberg numbering, minimal effective programming system, Rogers semilattice

1 Introduction

Let $\mathbb{N}$ be the set of natural numbers, i.e., $\{0, 1, 2, ...\}$. An effective programming systems (eps) is a partial computable function $\lambda p, x \,.\, \psi_p(x)$ mapping $\mathbb{N}^2$ to $\mathbb{N}$, and having the following property. For each partial computable function ζ mapping $\mathbb{N}$ to $\mathbb{N}$, there exists a p such that $\psi_p = \zeta$. Effective programming systems abstract the notion of *programming language* in the following sense. One can think of p as a *program*, and of ψ_p as the partial computable function denoted by p within some programming language corresponding to ψ.

Rogers [Rog58] introduced the following ordering on epses. For epses θ and ψ, $\theta \leq \psi$ iff there exists a computable function $t : \mathbb{N} \to \mathbb{N}$ such that, for each p, $\theta_p = \psi_{t(p)}$. Intuitively, $\theta \leq \psi$ whenever θ-programs can be algorithmically translated into ψ-programs. Moreover, an eps ψ is *minimal* in this ordering iff having the ability to algorithmically translate θ-programs into ψ-programs implies having the ability to algorithmically translate ψ-programs into θ-programs, for each eps θ.

Arguably, the most well studied collection of minimal epses is that of the Friedberg numberings [Fri58, Kum90]. Recall that a Friedberg numbering is an eps that is 1-1, i.e., for each p and q, $\psi_p = \psi_q$ implies $p = q$. Examples of works that make use of this concept include [Lav77, MWY78, Ric81, FKW82, Sch82, Roy87, Kum89, Spr90, GYY93, HK94, JST11].

In [PE64], Pour-El asked whether every minimal eps is equivalent to some Friedberg numbering. Ershov [Ers68, §5] showed that there exists a minimal effective numbering of the *computably enumerable sets* that is not equivalent to any 1-1 numbering. Shortly thereafter, his student, Khutoretskii, established the analogous result for the partial computable functions, thereby answering Pour-El's question.

Theorem 1 (Khutoretskii [Khu69a, Ex. 1 and Cor. 4]). There exists a minimal eps that is *not* equivalent to any Friedberg numbering.

For the purposes of this paper, Theorem 1 is best viewed through the following folklore theorem. (For completeness we give a proof of this result.)

Theorem 2 (Folklore). For each eps ψ, ψ is equivalent to a Friedberg numbering iff ψ's program equivalence relation is computable.

* This is an expanded version of [Moe12].

Proof. Let ψ be given.

($\Rightarrow$) Suppose that ψ is equivalent to a Friedberg numbering η, and that $t : \mathbb{N} \to \mathbb{N}$ witnesses $\psi \leq \eta$. Then, clearly, for each p and q,

$$\psi_p = \psi_q \;\Leftrightarrow\; \eta_{t(p)} = \eta_{t(q)} \;\Leftrightarrow\; t(p) = t(q). \tag{1}$$

Thus, since $\lambda p, q \centerdot [t(p) = t(q)]$ is computable, ψ's program equivalence relation is computable.

($\Leftarrow$) Suppose that ψ's program equivalence relation is computable. Let M be the set of *minimal programs* in ψ, i.e., $M = \{m_0, m_1, ...\}$ where, for each i, m_i is *least* such that

$$\psi_{m_i} \notin \{\psi_{m_0}, ..., \psi_{m_{i-1}}\}. \tag{2}$$

Note that, since ψ's program equivalence relation is computable, M is computable. Let η be such that, for each i,

$$\eta_i = \psi_{m_i}. \tag{3}$$

Using the fact the M is computable, it is straightforward to verify that η is a Friedberg numbering, and that $\psi \equiv \eta$. $\square$ (**Theorem 2**)

In light of Theorem 2, Theorem 1 may be restated as: there exists a minimal eps whose program equivalence relation is *not* computable. On the other hand, as noted in the proof of Theorem 1, the constructed eps's program equivalence relation is computably enumerable. (In particular, exactly one such equivalence class is a simple set [Rog67, §8.1], and all others a singletons.) Thus, one has the following.

Theorem 3 (Khutoretskii, corollary of Thm. 2 and proof of Thm. 1). There exists an eps whose program equivalence relation is computably enumerable, but *not* computable.

Subsequent to the above, Khutoretskii showed the following.

Theorem 4 (Khutoretskii, corollary of [Khu69b, Thm. 1]). There exists a minimal eps whose program equivalence relation is *not* computably enumerable.

Clearly, Theorems 3 and 4 can be viewed as a sharpening of Theorem 1. Herein, we sharpen Khutoretskii's results even further.

To facilitate the statement of our results, we first give a few definitions. Suppose that ψ is an eps. For each $t : \mathbb{N} \to \mathbb{N}$, we say that t is a *translation function into* ψ iff there exists an eps θ such that t witnesses $\theta \leq \psi$. The following definition is equivalent. For each $t : \mathbb{N} \to \mathbb{N}$, t is a translation function into ψ iff t is computable and the partial function $\lambda p, x \centerdot \psi_{t(p)}(x)$ is an eps.

Definition 5. Suppose that ψ is an eps, and that t is a translation function into ψ. Then, for each equivalence relation R, (a) and (b) below.

(a) R *strongly ties* t *into* ψ iff R satisfies (i) and (ii) just below.[1]
  (i) R is a subrelation of ψ's program equivalence relation.
  (ii) The range of t intersects each of R's equivalence classes.
(b) R *weakly ties* t *into* ψ iff R satisfies (i) just above and (ii*) just below.[2]
  (ii*) The range of t intersects all but finitely many of R's equivalence classes.

Thus, if equivalence relation R strongly ties translation function t into eps ψ, then R equates each ψ-program to some program in the range of t. If R merely weakly ties t into ψ, then there may be infinitely many ψ-programs that R does *not* equate to any program in the range of t. However, those infinitely many such ψ-programs will form only finitely many equivalence classes.

Our first main result is that the minimal epses may be *characterized* as follows.

Theorem 6. For each eps ψ, (a)-(c) below are equivalent.

(a) ψ is minimal.

[1] In some places, we omit the phrase "into ψ" when it is clear from context.
[2] See footnote 1.

$$P_0 \underset{\substack{\text{Thm. 2}\\\text{(Folklore)}}}{\Longleftrightarrow} P_0' \underset{\substack{\text{Thm. 3}\\\text{(Khutoretskii)}}}{\overset{\Longrightarrow}{\not\Longleftarrow}} P_1 \underset{\text{Thm. 9}}{\overset{\Longrightarrow}{\not\Longleftarrow}} P_2 \underset{\text{Thm. 8}}{\overset{\Longrightarrow}{\not\Longleftarrow}} P_3 \underset{\text{Thm. 7}}{\overset{\Longrightarrow}{\not\Longleftarrow}} P_4 \Longleftrightarrow \underset{\text{Thm. 6}}{P_4'} \Longleftrightarrow P_4''.$$

- $P_0(\psi) \Leftrightarrow \psi$ is equivalent to a Friedberg numbering.
- $P_0'(\psi) \Leftrightarrow \psi$'s program equivalence relation is computable.
- $P_1(\psi) \Leftrightarrow \psi$'s program equivalence relation is computably enumerable.
- $P_2(\psi) \Leftrightarrow$ there exists a ceer R that strongly ties each translation function into ψ.
- $P_3(\psi) \Leftrightarrow$ there exists a ceer R that weakly ties each translation function into ψ.
- $P_4(\psi) \Leftrightarrow$ for each translation function t into ψ, there exists a ceer that strongly ties t into ψ.
- $P_4'(\psi) \Leftrightarrow$ for each translation function t into ψ, there exists a ceer that weakly ties t into ψ.
- $P_4''(\psi) \Leftrightarrow \psi$ is minimal.

Fig. 1. A summary of the results mentioned in Section 1. In addition to the above: Mal'cev [Mal65, Mal71] showed that $P_1 \Rightarrow P_4''$, and Khutoretskii [Khu69b] showed that $P_1 \not\Leftarrow P_4''$ (see Theorem 4).

(b) For each translation function t into ψ, there exists a computably enumerable equivalence relation (ceer)[3] that strongly ties t into ψ.
(c) For each translation function t into ψ, there exists a ceer that weakly ties t into ψ.

Note that Theorem 4 is about a *single* equivalence relation, i.e., the program equivalence relation of a certain eps, whereas Theorem 6 is about one equivalence relation *per* translation function into any given eps. Thus, one might ask: if ψ is a minimal eps, then might there always exist a *single* ceer that strongly ties each translation function into ψ? The answer, as it turns out, is *no*. In fact, as Theorem 7 below states, there need not even exist a single ceer that *weakly* ties each translation function into ψ.

Theorem 7. There exists an eps ψ satisfying (a) and (b) below.

(a) ψ is minimal.
(b) For each ceer R, there exists a translation function t into ψ such that R does *not* weakly tie t into ψ.

Continuing with this line of thought, one finds that the strong and weak notions of Definition 5 separate when one considers single equivalence relations.

Theorem 8. There exists an eps ψ and a ceer R satisfying (a) and (b) below.

(a) For each translation function t into ψ, R weakly ties t into ψ.
(b) For each ceer R', there exists a translation function t into ψ such that R' does *not* strongly tie t into ψ.

Clearly, if ψ is an eps, and ψ's program equivalence relation is computably enumerable, then there exists a single ceer R that strongly ties each translation function into ψ, i.e., R is ψ's program equivalence relation. Thus, one might ask: does the converse hold? Theorem 9, just below, establishes that it does *not*.

Theorem 9. There exists an eps ψ and a ceer R satisfying (a) and (b) below.

(a) For each translation function t into ψ, R strongly ties t into ψ.
(b) ψ's program equivalence relation is *not* computably enumerable.

Figure 1 summarizes the results mentioned in this section. The remainder of this paper is organized as follows. Section 2 covers preliminaries. Section 3 gives complete proofs of Theorems 6 through 9.

[3] We pronounce ceer like the first syllable of "series". Computably enumerable equivalence relations are of interest in their own right. Gao and Gerdes [GG01] give an excellent survey.

2 Preliminaries

Computability-theoretic concepts not covered below are treated in [Rog67].

Lowercase math-italic letters (e.g., i, p, x), with or without decorations, range over elements of $\mathbb{N}$, unless stated otherwise. Uppercase math-italic letters (e.g., I, P, X), with or without decorations, range over subsets of $\mathbb{N}$, unless stated otherwise. For each *non*-empty X, $\min X$ denotes the minimum element of X. $\min \emptyset \stackrel{\text{def}}{=} \infty$. For each *non*-empty, finite X, $\max X$ denotes the maximum element of X. $\max \emptyset \stackrel{\text{def}}{=} -1$. $\mathcal{Fin}$ denotes the collection of all finite subsets of $\mathbb{N}$.

$\langle \cdot, \cdot \rangle$ denotes any fixed pairing function, i.e., a 1-1, onto, computable function of type $\mathbb{N}^2 \to \mathbb{N}$ [Rog67, page 64]. For each x, y, and z, $\langle x, y, z \rangle \stackrel{\text{def}}{=} \langle x, \langle y, z \rangle \rangle$. For each X and Y, $X \times Y \stackrel{\text{def}}{=} \{\langle x, y \rangle \mid x \in X \wedge y \in Y\}$.

Every partial function considered herein maps $\mathbb{N}$ to $\mathbb{N}$, unless stated otherwise. For each partial function ζ, and each x, $\zeta(x)\downarrow$ denotes that $\zeta(x)$ converges; whereas, $\zeta(x)\uparrow$ denotes that $\zeta(x)$ diverges. We use $\uparrow$ to denote the value of a divergent computation. For the sake of some subsequent proofs, it is convenient to have the following notation. For each i and n,

$$i^{<n} \stackrel{\text{def}}{=} \lambda x \centerdot \begin{cases} i, \text{ if } x < n; \\ \uparrow, \text{ otherwise.} \end{cases} \tag{4}$$

Thus, $i^{<n}$ is the partial function that maps each value less than n to i, and that diverges everywhere else. For each partial function ζ, $\mathrm{rng}(\zeta)$ denotes the range of ζ, i.e., $\mathrm{rng}(\zeta) \stackrel{\text{def}}{=} \{y \mid (\exists x)[\zeta(x) = y]\}$. $\mathcal{PartComp}$ denotes the set of all partial computable functions (mapping $\mathbb{N}$ to $\mathbb{N}$).

φ denotes any fixed acceptable (i.e., maximal) eps [Rog58, Rog67, MWY78, Ric81, Roy87]. For each p, $W_p \stackrel{\text{def}}{=} \{x \mid \varphi_p(x)\downarrow\}$. For each p and s, the following.

$$\varphi_p^s \stackrel{\text{def}}{=} \lambda x \centerdot \begin{cases} \varphi_p(x), & \text{if } x < s \text{ and } \varphi_p(x) \text{ converges in fewer than } s \text{ steps;} \\ \uparrow, & \text{otherwise.} \end{cases} \tag{5}$$

$$W_p^s \stackrel{\text{def}}{=} \{x \mid \varphi_p^s(x)\downarrow\}. \tag{6}$$

For each eps ψ, $\mathrm{Equiv}(\psi)$ denotes ψ's program equivalence relation, i.e.,

$$\mathrm{Equiv}(\psi) \stackrel{\text{def}}{=} \{\langle p, q \rangle \mid \psi_p = \psi_q\}. \tag{7}$$

For each equivalence relation R, $\mathcal{Classes}(R)$ denotes the set of R's equivalence classes, i.e., $\mathcal{Classes}(R)$ is the set of exactly those E satisfying (a)-(c) below.

(a) $E \neq \emptyset$.
(b) $(\forall p, q \in E)[\langle p, q \rangle \in R]$.
(c) $(\forall p \in E)(\forall q \notin E)[\langle p, q \rangle \notin R]$.

3 Results

This section recounts our main results (Theorem 6 through 9), and gives their complete proofs.

Our first main result is that the minimal epses may be *characterized* as per Theorem 6, restated just below. Recall from Definition 5 that if equivalence relation R strongly ties translation function t into eps ψ, then (i) R is a subrelation of ψ's program equivalence relation, and (ii) the range of t intersects each of R's equivalence classes. On the other hand, if R merely weakly ties t into ψ, then the range of t need only intersect all but finitely many of R's equivalence classes.

Theorem 6. For each eps ψ, (a)-(c) below are equivalent.

(a) ψ is minimal.
(b) For each translation function t into ψ, there exists a ceer that strongly ties t into ψ.
(c) For each translation function t into ψ, there exists a ceer that weakly ties t into ψ.

Proof. Let ψ be given.

(a) $\Rightarrow$ (b): Suppose that ψ is minimal. Let t be any translation function into ψ, and let θ be such that t witnesses $\theta \leq \psi$. Since ψ is minimal, there exists a $t' : \mathbb{N} \to \mathbb{N}$ witnessing $\psi \leq \theta$. Let R be the reflexive, symmetric, transitive closure of

$$\{\langle p, (t \circ t')(p)\rangle \mid p \in \mathbb{N}\}. \tag{8}$$

Clearly, R is a ceer and $R \subseteq \mathrm{Equiv}(\psi)$. It remains to show that, for each $E \in \mathcal{Classes}(R)$, $\mathrm{rng}(t) \cap E \neq \emptyset$. So, let $E \in \mathcal{Classes}(R)$ be given, and let $p \in E$ be arbitrary. Then, clearly, $(t \circ t')(p) \in \mathrm{rng}(t) \cap E$.

(b) $\Rightarrow$ (c): Immediate.

(c) $\Rightarrow$ (a): Suppose (c). Further suppose that θ is an eps, and that $t : \mathbb{N} \to \mathbb{N}$ witnesses $\theta \leq \psi$. Then, by (c), there exists a ceer $R \subseteq \mathrm{Equiv}(\psi)$ such that, for all but finitely many $E \in \mathcal{Classes}(R)$, $\mathrm{rng}(t) \cap E \neq \emptyset$. Let n be the number of elements of $\mathcal{Classes}(R)$ that do *not* intersect $\mathrm{rng}(t)$, and let $E_0, ..., E_{n-1}$ be those elements. Choose $q_0, ..., q_{n-1}$ such that, for each $i < n$ and $p \in E_i$, $\theta_{q_i} = \psi_p$. Note that, for each p, either R equates p to some element of $\mathrm{rng}(t)$, or $p \in E_i$, for some $i < n$. It follows that the function $t' : \mathbb{N} \to \mathbb{N}$, defined next, is computable.

$$t' = \lambda p\centerdot \begin{cases} q, & \text{where } q \text{ is first found such that } \langle p, t(q)\rangle \in R, \\ & \quad \text{if such a } q \text{ exists;} \\ q_i, & \text{otherwise, where } i \text{ is such that } p \in E_i. \end{cases} \tag{9}$$

It is straightforward to verify that t' witnesses $\psi \leq \theta$. $\square$ (**Theorem 6**)

Theorem 7, restated just below, is our second main result. It establishes that there there exists a minimal eps ψ such that, for each ceer R, either R contradicts ψ's program equivalence relation, or there exists a translation function t into ψ such that the range of t *fails* to intersect *infinitely many* of R's equivalence classes.

Theorem 7. There exists an eps ψ satisfying (a) and (b) below.

(a) ψ is minimal.
(b) For each ceer R, there exists a translation function t into ψ such that R does *not* weakly tie t into ψ.

The proof of Theorem 7 makes use of the following lemma.

Lemma 10. Let $J_0, ..., J_{n-1}$ be any finite collection of computably enumerable sets. Then, there exists an infinite, computable set X, and a finite set $L \subseteq \{0, ..., n-1\}$, such that, for each $x \in X$ and $\ell < n$, $x \in J_\ell$ iff $\ell \in L$.

Proof. Let $J_0, ..., J_{n-1}$ be as stated. The set X is the set X_n, constructed as follows. Set $X_0 = \mathbb{N}$. Then, for each $\ell < n$, act according to the following conditions.

– Cond. (a) [$J_\ell \cap X_\ell$ is infinite]. Set $X_{\ell+1}$ to any infinite, computable subset of $J_\ell \cap X_\ell$.
– Cond. (b) [$J_\ell \cap X_\ell$ is finite]. Set $X_{\ell+1} = \{x \in X_\ell \mid x > \max(J_\ell \cap X_\ell)\}$.

The set L is such that

$$L = \{\ell \mid \text{cond. (a) applies for } \ell\}. \tag{10}$$

Clearly, X is infinite and computable. Further note that

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n. \tag{11}$$

It is easily seen that, for each $\ell < n$: if $\ell \in L$, then $J_\ell \supseteq X_{\ell+1}$; whereas, if $\ell \notin L$, then $J_\ell \cap X_{\ell+1} = \emptyset$. It then follows from (11) that, for each $x \in X_n$ and $\ell < n$, $x \in J_\ell$ iff $\ell \in L$. $\square$ (**Lemma 10**)

Proof of Theorem 7. The eps ψ is constructed below, following some necessary definitions. Let $\mathcal{Aux} \subseteq \mathcal{PartComp}$ be such that

$$\mathcal{Aux} = \mathcal{PartComp} \setminus \{\langle i, j\rangle^{<k+1} \mid i, j \in \mathbb{N} \ \wedge \ k < 2^i\}. \tag{12}$$

It is straightforward to show that $\mathcal{Aux}$ is 1-1, computably enumerable. So, let $(\alpha_\ell)_{\ell \in \mathbb{N}}$ be a 1-1, effective numbering of $\mathcal{Aux}$.

As is common, ψ is constructed in stages, i.e., ψ is the union of $\psi^0 \subseteq \psi^1 \subseteq \cdots$. In conjunction with ψ, four computable predicates are constructed: $\lambda i, s\centerdot[i \in R\text{-flags}^s]$, $\lambda i, j, \ell, s\centerdot[\langle i, j, \ell\rangle \in t\text{-flags}^s]$, $\lambda \ell, s\centerdot[\ell \in \mathrm{Src}^s]$, and $\lambda p, s\centerdot[p \in \mathrm{Dst}^s]$. The purposes of these predicates are as follows.

- The R-flags predicate keeps track of which i are such that W_i contradicts ψ's program equivalence relation. More precisely, for each i, if there exists an s such that $i \in R\text{-flags}^s$, then $W_i \not\subseteq \mathrm{Equiv}(\psi)$.
- The t-flags predicate helps to keep track of which ℓ *may be* such that φ_ℓ is a translation function into ψ. It will turn out that: if i and ℓ are such that $W_i \subseteq \mathrm{Equiv}(\psi)$ and φ_ℓ is a translation function into ψ, then, for each j, and all but finitely many s, $\langle i, j, \ell\rangle \in t\text{-flags}^s$.
- The Src predicate keeps track of which ℓ are such that α_ℓ has not yet been assigned to any ψ-program. In particular, if ℓ and s are such that $\ell \in \mathrm{Src}^s$ and $\alpha_\ell \neq \lambda x.\!\uparrow$, then, for each p, $\psi_p^s \neq \alpha_\ell$.
- The Dst predicate keeps track of which ψ-programs have not yet been used. More precisely, if p and s are such that $p \in \mathrm{Dst}^s$, then $\psi_p^s = \lambda x.\!\uparrow$.

For each i and s, $i \in R\text{-flags}^{s+1}$ iff $i \in R\text{-flags}^s$, unless stated otherwise. Analogous statements apply to the t-flags, Src, and Dst predicates, as well. The following will be clear from the construction of ψ, for each s.

$$R\text{-flags}^s \subseteq R\text{-flags}^{s+1}. \tag{13}$$
$$t\text{-flags}^s \subseteq t\text{-flags}^{s+1}. \tag{14}$$
$$\mathrm{Src}^s \supseteq \mathrm{Src}^{s+1}. \tag{15}$$
$$\mathrm{Dst}^s \supseteq \mathrm{Dst}^{s+1}. \tag{16}$$

Let $\mathrm{height} : \mathbb{N}^3 \to \mathbb{N}$ be such that, for each i, j, and s,

$$\mathrm{height}_{i,j}^s = |\{\ell \mid \langle i, j, \ell\rangle \in t\text{-flags}^s\}|. \tag{17}$$

It will be clear from the construction of ψ that, for each i, j, ℓ, and s,

$$\langle i, j, \ell\rangle \in t\text{-flags}^s \;\Rightarrow\; \ell < i. \tag{18}$$

Thus, for each i, j, and s,

$$\mathrm{height}_{i,j}^s \leq i. \tag{19}$$

Let $\mathrm{num} : \mathbb{N}^3 \to \mathbb{N}$ be such that, for each i, j, and s,

$$\mathrm{num}_{i,j}^s = 2^{i-h}, \text{ where } h = \mathrm{height}_{i,j}^s. \tag{20}$$

Let $f : \mathbb{N}^3 \to \mathbb{N}$ be such that, for each i, j, and k,

$$f_{i,j}(k) = 2\langle i, j \cdot 2^{i+1} + k\rangle. \tag{21}$$

For each i, j, s, and $k < \mathrm{num}_{i,j}^s$, let $E_{i,j,k}^s \in \mathcal{F}\!in$ and $\bar{E}_{i,j,k}^s \in \mathcal{F}\!in$ be as follows, with $h = \mathrm{height}_{i,j}^s$.

$$E_{i,j,k}^s = \big\{f_{i,j}\big(k \cdot 2^{h+1}\big), ..., f_{i,j}\big(k \cdot 2^{h+1} + 2^h - 1\big)\big\}. \tag{22}$$
$$\bar{E}_{i,j,k}^s = \big\{f_{i,j}\big(k \cdot 2^{h+1} + 2^h\big), ..., f_{i,j}\big((k+1) \cdot 2^{h+1} - 1\big)\big\}. \tag{23}$$

Note that, for each i, j, and s, if one lets $h = \mathrm{height}_{i,j}^s$, and it happens that $\mathrm{height}_{i,j}^{s+1} = h + 1$, then, for each $k < \mathrm{num}_{i,j}^{s+1}$,

$$\begin{aligned}
E_{i,j,k}^{s+1} &= \{f_{i,j}(k \cdot 2^{h+2}), ..., f_{i,j}(k \cdot 2^{h+2} + 2^{h+1} - 1)\} \\
&= \quad \{f_{i,j}(k \cdot 2^{h+2}), ..., f_{i,j}(k \cdot 2^{h+2} + 2^h - 1)\} \\
&\quad \cup \{f_{i,j}(k \cdot 2^{h+2} + 2^h), ..., f_{i,j}(k \cdot 2^{h+2} + 2^{h+1} - 1)\} \\
&= \quad \{f_{i,j}(2k \cdot 2^{h+1}), ..., f_{i,j}(2k \cdot 2^{h+1} + 2^h - 1)\} \\
&\quad \cup \{f_{i,j}(2k \cdot 2^{h+1} + 2^h), ..., f_{i,j}(2k \cdot 2^{h+1} + 2^{h+1} - 1)\} \\
&= \quad \big\{f_{i,j}\big(2k \cdot 2^{h+1}\big), ..., f_{i,j}\big(2k \cdot 2^{h+1} + 2^h - 1\big)\big\} \\
&\quad \cup \big\{f_{i,j}\big(2k \cdot 2^{h+1} + 2^h\big), ..., f_{i,j}\big((2k+1) \cdot 2^{h+1} - 1\big)\big\} \\
&= E_{i,j,2k}^s \cup \bar{E}_{i,j,2k}^s.
\end{aligned} \tag{24}$$

It can be shown that, under the same conditions,

$$\bar{E}_{i,j,k}^{s+1} = E_{i,j,2k+1}^s \cup \bar{E}_{i,j,2k+1}^s. \tag{25}$$

– Stage $s = -1$. Do the following.
  - Set $R\text{-flags}^0 = \emptyset$.
  - Set $t\text{-flags}^0 = \emptyset$.
  - Set $\mathrm{Src}^0 = \mathbb{N}$.
  - Set $\mathrm{Dst}^0 = 2\mathbb{N} + 1$.
  - For each i, j, and $k < 2^i$, set $\psi^0_{f_{i,j}(2k)} = \psi^0_{f_{i,j}(2k+1)} = \langle i, j\rangle^{<k+1}$.
  - For each $p \in 2\mathbb{N} + 1$, set $\psi^0_p = \lambda x \centerdot \uparrow$.

– Stage $s = \langle 0, \ell\rangle$. If $\ell \in \mathrm{Src}^s$, then do the following.
  - Set $\mathrm{Src}^{s+1} = \mathrm{Src}^s \setminus \{\ell\}$.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \{\min \mathrm{Dst}^s\}$.
  - Set $\psi^{s+1}_{\min \mathrm{Dst}^s} = \alpha_\ell$.

– Stage $s = \langle i + 1, 0, -\rangle$. Determine whether there exist j and k satisfying conditions (a)-(c) just below.
  (a) $i \notin R\text{-flags}^s$.
  (b) $k < \mathrm{num}^s_{i,j}$.
  (c) $W^s_i \cap (E^s_{i,j,k} \times \bar{E}^s_{i,j,k}) \neq \emptyset$.
  If such j and k exist, then do the following.
  - Set $R\text{-flags}^{s+1} = R\text{-flags}^s \cup \{i\}$.
  - Choose any $\ell, m \in \mathrm{Src}^s$ such that $\ell \neq m$ and $\langle i, j\rangle^{<2^i} \subseteq \alpha_\ell \cap \alpha_m$.
  - Let $d : \mathbb{N} \to \mathbb{N}$ be any 1-1, computable function such that $\mathrm{rng}(d)$ is computable, $\mathrm{rng}(d) \subseteq \mathrm{Dst}^s$, and $\mathrm{Dst}^s \setminus \mathrm{rng}(d)$ is infinite.
  - Set $\mathrm{Src}^{s+1} = \mathrm{Src}^s \setminus \{\ell, m\}$.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \mathrm{rng}(d)$.
  - For each j, each $k < \mathrm{num}^s_{i,j}$, and each $p \in E_{i,j,k}$, set $\psi^{s+1}_p = \alpha_\ell$.
  - For each j, each $k < \mathrm{num}^s_{i,j}$, and each $q \in \bar{E}_{i,j,k}$, set $\psi^{s+1}_q = \alpha_m$.
  - For each j and $k < \mathrm{num}^s_{i,j}$, set $\psi^{s+1}_{d(n+k)} = \langle i, j\rangle^{<(k+1)\cdot 2^h}$, where $n = \sum_{\hat{\jmath}<j} \mathrm{num}^s_{i,\hat{\jmath}}$ and $h = \mathrm{height}^s_{i,j}$.

– Stage $s = \langle i + 1, j + 1, \ell, -\rangle$. Let $h = \mathrm{height}^s_{i,j}$. Determine whether conditions (i)-(iv) just below are satisfied.
  (i) $\ell < i$.
  (ii) $i \notin R\text{-flags}^s$.
  (iii) $\langle i, j, \ell\rangle \notin t\text{-flags}^s$.
  (iv) For each $k < \mathrm{num}^s_{i,j}$, $\mathrm{rng}(\varphi^s_\ell) \cap (E^s_{i,j,k} \cup \bar{E}^s_{i,j,k}) \neq \emptyset$.
  If so, then do the following.
  - Set $t\text{-flags}^{s+1} = t\text{-flags}^s \cup \{\langle i, j, \ell\rangle\}$. (Note that this implies $\mathrm{height}^{s+1}_{i,j} = \mathrm{height}^s_{i,j} + 1$.)
  - Let $n = \mathrm{num}^{s+1}_{i,j}$. (Note that, by the just previous step, $n = \mathrm{num}^s_{i,j}/2$.)
  - Let $\{q_0 < q_1 < \cdots < q_{n-1}\}$ be the n *least* elements of Dst^s.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \{q_0, q_1, ..., q_{n-1}\}$.
  - For each $k < n$ and $p \in (E^{s+1}_{i,j,k} \cup \bar{E}^{s+1}_{i,j,k})$, set $\psi^{s+1}_p = \langle i, j\rangle^{(2k+2)\cdot 2^h}$.
  - For each $k < n$, set $\psi^{s+1}_{q_k} = \langle i, j\rangle^{<(2k+1)\cdot 2^h}$.

Fig. 2. The construction of ψ in the proof of Theorem 7. The symbols height, num, f, E, and $\bar{E}$ are defined in (17), (20), (21), (22), and (23), respectively.

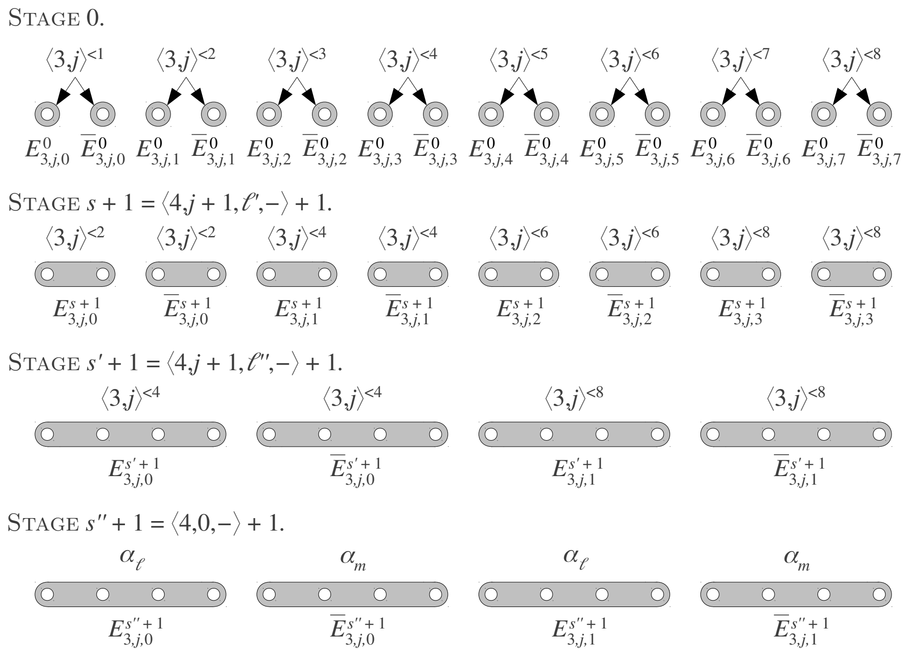

Fig. 3. A depiction of what *could* happen in the proof of Theorem 7 with respect to the ψ-programs of the form $f_{3,j}(k)$, where j is arbitrary and $k < 16$ (see text).

The partial function ψ is constructed in Figure 2. To help to give some of the intuition behind the construction, Figure 3 depicts what *could* happen with respect to the ψ-programs of the form $f_{3,j}(k)$, where j is arbitrary and $k < 16$. In stage 0, the programs will form *eight* pairs of equivalence classes, where the kth pair computes $\langle 3, j\rangle^{<k+1}$ (the first such pair being the 0th). If, subsequently, the conditions of some stage s of the form $\langle 4, j+1, \ell', -\rangle$ are satisfied, then, in stage $s+1$, the programs will form *four* pairs of equivalence classes, where the kth pair computes $\langle 3, j\rangle^{<2k+2}$. If, similarly, the conditions of some stage s' of the form $\langle 4, j+1, \ell'', -\rangle$ are satisfied (where $\ell' \neq \ell''$), then, in stage $s'+1$, the programs will form *two* pairs of equivalence classes, where the kth pair computes $\langle 3, j\rangle^{<4k+4}$. If, finally, the conditions of some stage s'' of the form $\langle 4, 0, -\rangle$ are satisfied, then, in stage $s''+1$, the equivalence classes will alternate in computing α_ℓ and α_m, for some *distinct* ℓ and m.

Note that by (14), (17), and (19), the following function $\text{height}^\infty : \mathbb{N}^2 \to \mathbb{N}$ is well-defined. For each i and j,

$$\text{height}^\infty_{i,j} = \max\{\text{height}^s_{i,j} \mid s \in \mathbb{N}\}. \tag{26}$$

For each i and j, let $\text{num}^\infty_{i,j}$ be defined in a manner analogous to (20), but with $h = \text{height}^\infty_{i,j}$. For each i, j, and $k < \text{num}^\infty_{i,j}$, let $E^\infty_{i,j,k}$ and $\bar{E}^\infty_{i,j,k}$ be defined in a manner analogous to (22) and (23) (respectively), but with $h = \text{height}^\infty_{i,j}$.

Claim 7.1 below establishes that ψ is an eps. Claim 7.7 below establishes that ψ satisfies (a) in the statement of the theorem, i.e., that ψ is minimal. Claim 7.8 below establishes that ψ satisfies (b) in the statement of the theorem, i.e., that for each ceer R, there exists a translation function t into ψ such that R does *not* weakly tie t into ψ.

Claim 7.1. ψ is an eps.

Proof of Claim. Clearly, ψ is partial computable. Thus, it suffices to show that, for each $\zeta \in \mathcal{PartComp}$, there exists a p such that $\psi_p = \zeta$. So, let $\zeta \in \mathcal{PartComp}$ be given. Consider the following cases.

Case [$\zeta \in \mathcal{Aux}$]. Let ℓ be such that $\alpha_\ell = \zeta$, and let $s = \langle 0, \ell\rangle$. Then, the following are easily verifiable from the construction of ψ.

– If $\ell \notin \text{Src}^s$, then there exists a p of the form $f_{i,j}(k)$, for some i, j, and k, such that $\psi^s_p = \zeta$.

– If $\ell \in \mathrm{Src}^s$, then $\psi^{s+1}_{\min \mathrm{Dst}^s} = \zeta$.

Case [$\zeta \notin \mathcal{A}ux$]. Let i, j, k, and h be such that $\zeta = \langle i,j\rangle^{<(2k+1)\cdot 2^h}$. Then, the following are easily verifiable from the construction of ψ.

– If $\mathrm{height}^\infty_{i,j} \le h$ and $(\forall s)[i \notin R\text{-flags}^s]$, then, for each p,

$$p \in \big\{ f_{i,j}\big((2k+1)\cdot 2^{h+1} - 2\big), f_{i,j}\big((2k+1)\cdot 2^{h+1} - 1\big) \big\} \;\Rightarrow\; \psi_p = \zeta. \tag{27}$$

– If $\mathrm{height}^\infty_{i,j} > h$ or $(\exists s)[i \in R\text{-flags}^s]$, then there exists a $p \in \mathrm{Dst}^0$ $(= 2\mathbb{N}+1)$ such that $\psi_p = \zeta$.

□ (**Claim 7.1**)

Claim 7.2. Suppose that i is such that $(\forall s)[i \notin R\text{-flags}^s]$. Then, for each j, each $k < \mathrm{num}^\infty_{i,j}$, and each p,

$$p \in (E^\infty_{i,j,k} \cup \bar{E}^\infty_{i,j,k}) \;\Leftrightarrow\; \psi_p = \langle i,j\rangle^{(k+1)\cdot 2^h}, \tag{28}$$

where $h = \mathrm{height}^\infty_{i,j}$.

Proof of Claim. Easily verifiable from the construction of ψ. □ (**Claim 7.2**)

Claim 7.3. Suppose that i is such that $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\min}$ be *least* such that $i \in R\text{-flags}^{s_{\min}+1}$. Then, there exist *distinct* ℓ and m such that (a) and (b) below.

(a) For each p,

$$p \in \bigcup\{E^{s_{\min}}_{i,j,k} \mid j \in \mathbb{N} \;\wedge\; k < \mathrm{num}^{s_{\min}}_{i,j}\} \;\Leftrightarrow\; \psi_p = \alpha_\ell. \tag{29}$$

(b) For each q,

$$q \in \bigcup\{\bar{E}^{s_{\min}}_{i,j,k} \mid j \in \mathbb{N} \;\wedge\; k < \mathrm{num}^{s_{\min}}_{i,j}\} \;\Leftrightarrow\; \psi_q = \alpha_m. \tag{30}$$

Proof of Claim. Easily verifiable from the construction of ψ. □ (**Claim 7.3**)

Claim 7.4. For each $p \in \mathrm{Dst}^0$ $(= 2\mathbb{N}+1)$ and q, if $\psi_p = \psi_q$, then $p = q$.

Proof of Claim. Easily verifiable from the construction of ψ. □ (**Claim 7.4**)

Claim 7.5. Suppose that i, j, ℓ, and s are such that $\langle i,j,\ell\rangle \in t\text{-flags}^s$. Then,

$$\mathrm{rng}(\varphi_\ell) \cap E^s_{i,j,k} \neq \emptyset \;\wedge\; \mathrm{rng}(\varphi_\ell) \cap \bar{E}^s_{i,j,k} \neq \emptyset. \tag{31}$$

Proof of Claim. Suppose that i, j, ℓ, and s are as stated. Let $s_{\min}$ be *least* such that

$$\langle i,j,\ell\rangle \in t\text{-flags}^{s_{\min}+1}. \tag{32}$$

Thus, $s > s_{\min}$. By the construction of ψ, for each $k' < \mathrm{num}^{s_{\min}}_{i,j}$,

$$\mathrm{rng}(\varphi_\ell) \cap (E^{s_{\min}}_{i,j,k'} \cup \bar{E}^{s_{\min}}_{i,j,k'}) \neq \emptyset. \tag{33}$$

It follows from (24) and (32) that, for each $s > s_{\min}$ and $k < \mathrm{num}^s_{i,j}$, there exists a $k' < \mathrm{num}^{s_{\min}}_{i,j}$ such that

$$E^{s_{\min}}_{i,j,k'} \cup \bar{E}^{s_{\min}}_{i,j,k'} \subseteq E^s_{i,j,k}. \tag{34}$$

Similarly, it follows from (25) and (32) that, for each $s > s_{\min}$ and $k < \mathrm{num}^s_{i,j}$, there exists a $k' < \mathrm{num}^{s_{\min}}_{i,j}$ such that

$$E^{s_{\min}}_{i,j,k'} \cup \bar{E}^{s_{\min}}_{i,j,k'} \subseteq \bar{E}^s_{i,j,k}. \tag{35}$$

Formula (31) is implied by (33), (34), and (35). □ (**Claim 7.5**)

For each i, j, and s, act according to the following computable conditions. (Note that cond. (a) is computable, in part, because there are only finitely many $i \le \ell$.)

- Cond. (a) $\big[\text{height}^s_{i,j} < \text{height}^{s+1}_{i,j} \ \wedge\ i \le \ell \ \wedge\ (\forall s)[i \notin R\text{-flags}^s]\big]$. For each $k < \text{num}^{s+1}_{i,j}$ and
$$p, q \in (E^{s+1}_{i,j,k} \cup \bar{E}^{s+1}_{i,j,k}),$$
list $\langle p, q\rangle$ into R.
- Cond. (b) $\big[\text{height}^s_{i,j} < \text{height}^{s+1}_{i,j} \ \wedge\ i > \ell\big]$. For each $k < \text{num}^{s+1}_{i,j}$ and
$$p, q \in E^{s+1}_{i,j,k},$$
list $\langle p, q\rangle$ into R. Similarly, for each
$$p, q \in \bar{E}^{s+1}_{i,j,k},$$
list $\langle p, q\rangle$ into R.

For each i, act according to the following partial computable condition.

- Cond. (c) $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\min}$ be *least* such that $i \in R\text{-flags}^{s_{\min}+1}$, and do the following. For each
$$p, q \in \bigcup\{E^{s_{\min}}_{i,j,k} \mid j \in \mathbb{N} \ \wedge\ k < \text{num}^{s_{\min}}_{i,j}\},$$
list $\langle p, q\rangle$ into R. Similarly, for each
$$p, q \in \bigcup\{\bar{E}^{s_{\min}}_{i,j,k} \mid j \in \mathbb{N} \ \wedge\ k < \text{num}^{s_{\min}}_{i,j}\},$$
list $\langle p, q\rangle$ into R.

Fig. 4. The construction of R in the proof of Claim 7.7.

Claim 7.6. Suppose that i is such that $W_i \subseteq \text{Equiv}(\psi)$. Then, $(\forall s)[i \notin R\text{-flags}^s]$.

Proof of Claim. The proof is by contrapositive. Suppose that i is such that $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\min}$ be *least* such that $i \in R\text{-flags}^{s_{\min}+1}$. Then, by the construction of ψ, there exist j and k such that
$$W_i^{s_{\min}} \cap (E^{s_{\min}}_{i,j,k} \times \bar{E}^{s_{\min}}_{i,j,k}) \neq \emptyset. \tag{36}$$
Furthermore, by Claim 7.3($\Rightarrow$), there exist *distinct* ℓ and m such that (a) and (b) below.

(a) For each $p \in E^{s_{\min}}_{i,j,k}$, $\psi_p = \alpha_\ell$.
(b) For each $q \in \bar{E}^{s_{\min}}_{i,j,k}$, $\psi_q = \alpha_m$.

Since α is 1-1 and $\ell \neq m$, $\alpha_\ell \neq \alpha_m$. Thus, by (36) and (a) and (b) just above, $W_i \not\subseteq \text{Equiv}(\psi)$. $\square$ (**Claim 7.6**)

Claim 7.7. ψ satisfies (a) in the statement of the theorem, i.e., ψ is minimal.

Proof of Claim. Let t be any translation function into ψ, and let ℓ be such that $\varphi_\ell = t$. To show the claim, a ceer R is exhibited such that R strongly ties t into ψ. Initially, R consists of $\{\langle p, p\rangle \mid p \in \mathbb{N}\}$. Then, pairs are added to R as in Figure 4.

Clearly, R is a ceer. That $R \subseteq \text{Equiv}(\psi)$ follows from the ($\Rightarrow$) directions of Claims 7.2 and 7.3. It remains to show that, for each $E \in \mathit{Classes}(R)$, $\text{rng}(t) \cap E \neq \emptyset$. It is straightforward to verify that each $E \in \mathit{Classes}(R)$ is of one of the following four types.

- Type I. E is of the form
$$E^{\infty}_{i,j,k} \cup \bar{E}^{\infty}_{i,j,k}, \tag{37}$$
where: $i \le \ell$, $(\forall s)[i \notin R\text{-flags}^s]$, j is arbitrary, and $k < \text{num}^{\infty}_{i,j}$. (Intuitively, E is the result of one or more invocations of cond. (a) in Figure 4.)

– TYPE II. Either E is of the form

$$E^{\infty}_{i,j,k} \tag{38}$$

or E is of the form

$$\bar{E}^{\infty}_{i,j,k} \tag{39}$$

where: $i > \ell$, $(\forall s)[i \notin R\text{-flags}^s]$, j is arbitrary, and $k < \text{num}^{\infty}_{i,j}$. (Intuitively, E is the result of one or more invocations of cond. (b) in Figure 4.)

– TYPE III. Either E is of the form

$$\bigcup\{E^{s_{\min}}_{i,j,k} \mid j \in \mathbb{N} \ \wedge \ k < \text{num}^{s_{\min}}_{i,j}\} \tag{40}$$

or E is of the form

$$\bigcup\{\bar{E}^{s_{\min}}_{i,j,k} \mid j \in \mathbb{N} \ \wedge \ k < \text{num}^{s_{\min}}_{i,j}\} \tag{41}$$

where: i is such that $(\exists s)[i \in R\text{-flags}^s]$, and $s_{\min}$ is *least* such that $i \in R\text{-flags}^{s_{\min}+1}$. (Intuitively, E is the result of zero or more invocations of cond. (b) in Figure 4, followed by a single invocation of cond. (c).)

– TYPE IV. $E = \{p\}$, for some $p \in \text{Dst}^0$ $(= 2\mathbb{N}+1)$.

Let $E \in \mathcal{C}lasses(R)$ be given. If E is of type I, then it follows from Claim 7.2($\Leftarrow$) that $\text{rng}(t) \cap E \neq \emptyset$. If E is of type III, then it follows from Claim 7.3($\Leftarrow$) that $\text{rng}(t) \cap E \neq \emptyset$. If E is of type IV, then it follows from Claim 7.4 that $\text{rng}(t) \cap E \neq \emptyset$.

So, suppose that E is of type II. Let i, j, and k be such that $E = E^{\infty}_{i,j,k}$ or $\bar{E} = E^{\infty}_{i,j,k}$, as appropriate. Further suppose, by way of contradiction, that $\text{rng}(t) \cap E = \emptyset$. Thus,

$$\text{rng}(t) \cap E^{\infty}_{i,j,k} = \emptyset \ \vee \ \text{rng}(t) \cap \bar{E}^{\infty}_{i,j,k} = \emptyset. \tag{42}$$

Note that by Claim 7.2($\Leftarrow$), for each $k' < \text{num}^{\infty}_{i,j}$ and p,

$$\psi_p = \langle i,j \rangle^{(k'+1)\cdot 2^h} \ \Rightarrow \ p \in (E^{\infty}_{i,j,k'} \cup \bar{E}^{\infty}_{i,j,k'}), \tag{43}$$

where $h = \text{height}^{\infty}_{i,j}$. Thus, since t is a translation function into ψ, it must be the case that, for each $k' < \text{num}^{\infty}_{i,j}$,

$$\text{rng}(t) \cap (E^{\infty}_{i,j,k'} \cup \bar{E}^{\infty}_{i,j,k'}) \neq \emptyset. \tag{44}$$

Choose s such that s is of the form $\langle i+1, j+1, \ell, - \rangle$, $\text{height}^s_{i,j} = \text{height}^{\infty}_{i,j}$, and, for each $k' < \text{num}^{\infty}_{i,j}$,

$$\text{rng}(\varphi^s_\ell) \cap (E^{\infty}_{i,j,k'} \cup \bar{E}^{\infty}_{i,j,k'}) \neq \emptyset. \tag{45}$$

Note that by (42) and Claim 7.5, $\langle i,j,\ell \rangle \notin t\text{-flags}^s$. It follows that all of the conditions of stage s are satisfied. Thus, $\langle i,j,\ell \rangle \in t\text{-flags}^{s+1}$. But then

$$\text{height}^{s+1}_{i,j} > \text{height}^s_{i,j} = \text{height}^{\infty}_{i,j} \tag{46}$$

— a contradiction. □ (**Claim 7.7**)

Claim 7.8. ψ satisfies (b) in the statement of the theorem, i.e., for each ceer R, there exists a translation function t into ψ such that R does *not* weakly tie t into ψ.

Proof of Claim. Suppose that ceer R is such that

$$R \subseteq \text{Equiv}(\psi). \tag{47}$$

Let i be such that $W_i = R$. Note that by Claim 7.6,

$$(\forall s)[i \notin R\text{-flags}^s]. \tag{48}$$

For each $\ell < i$, let J_ℓ be as follows.

$$J_\ell = \{j \mid (\exists s)[\langle i,j,\ell \rangle \in t\text{-flags}^s]\}. \tag{49}$$

Clearly, for each $\ell < i$, J_ℓ is computably enumerable. Thus, by Lemma 10, there exists an infinite, computable set X, and a finite set $L \subseteq \{0, ..., i-1\}$, such that, for each $x \in X$ and $\ell \in L$, $x \in J_\ell$ iff $\ell \in L$. Thus, for each $x \in X$,

$$\begin{aligned} L &= \{\ell \mid x \in J_\ell\} \\ &= \big\{\ell \mid x \in \{j \mid (\exists s)[\langle i, j, \ell\rangle \in t\text{-flags}^s]\}\big\} \\ &= \{\ell \mid (\exists s)[\langle i, x, \ell\rangle \in t\text{-flags}^s]\}. \end{aligned}$$

It follows that, for each $x \in X$, $\text{height}^\infty_{i,x} = |L|$ and $\text{num}^\infty_{i,j} = 2^{i-|L|}$. Let t be any computable function such that

$$\text{rng}(t) = \mathbb{N} \setminus \bigcup\{E^\infty_{i,x,0} \mid x \in X\}.^4 \tag{50}$$

It is straightforward to show that that t is a translation function into ψ. On the other hand, it is clearly the case that, for each $x \in X$,

$$\text{rng}(t) \cap E^\infty_{i,x,0} = \emptyset. \tag{51}$$

Thus, to complete the proof, it suffices to show that, for each $E \in \text{Equiv}(R)$ and $x \in X$,

$$E \cap E^\infty_{i,x,0} \neq \emptyset \ \Rightarrow \ E \subseteq E^\infty_{i,x,0}. \tag{52}$$

By way of contradiction, suppose otherwise, as witnessed by E and x, i.e.,

$$E \cap E^\infty_{i,x,0} \neq \emptyset \ \wedge \ E \not\subseteq E^\infty_{i,x,0}. \tag{53}$$

By (47), (48), (53), and Claim 7.2 (both directions), it must be the case that

$$E \cap \bar{E}^\infty_{i,x,0} \neq \emptyset., \tag{54}$$

Thus, by the first conjunct of (53), and by (54), there exists a stage s of the form $\langle i+1, 0, -\rangle$ in which all of the conditions of that stage are satisfied. Thus, $i \in R\text{-flags}^{s+1}$. But this contradicts (48). □ (**Claim 7.8**)

□ (**Theorem 7**)

Theorem 8, restated just below, is our third main result. It establishes that the strong and weak notions of Definition 5 separate when one considers single equivalence relations.

Theorem 8. There exists an eps ψ and a ceer $R \subseteq \text{Equiv}(\psi)$ satisfying (a) and (b) below.

(a) For each translation function t into ψ, R weakly ties t into ψ.
(b) For each ceer R', there exists a translation function t into ψ such that R' does *not* strongly tie t into ψ.

Proof. The proof is essentially a modification to the proof of Theorem 7. Intuitively, one eliminates all uses of j in that proof. So, for example, for each i, rather than start with infinitely many pairs of equivalence classes,

$$\{(E^0_{i,j,k}, \bar{E}^0_{i,j,k}) \mid j \in \mathbb{N} \ \wedge \ k < 2^i\}, \tag{55}$$

one instead starts with just 2^i many such pairs,

$$\{(E^0_{i,k}, \bar{E}^0_{i,k}) \mid k < 2^i\}. \tag{56}$$

This has the effect of invalidating Claim 7.8 (and of making Lemma 10 unnecessary).

Let $\mathcal{Aux} \subseteq \mathcal{PartComp}$ be such that

$$\mathcal{Aux} = \mathcal{PartComp} \setminus \{i^{<k+1} \mid i \in \mathbb{N} \ \wedge \ k < 2^i\}. \tag{57}$$

Let $(\alpha_\ell)_{\ell\in\mathbb{N}}$ be a 1-1, effective numbering of $\mathcal{Aux}$.

In conjunction with ψ, four computable predicates are constructed: $\lambda i, s.[i \in R\text{-flags}^s]$, $\lambda i, \ell, s.[\langle i, \ell\rangle \in t\text{-flags}^s]$, $\lambda \ell, s.[\ell \in \text{Src}^s]$, and $\lambda p, s.[p \in \text{Dst}^s]$. The purposes of these predicates are similar to those in the proof of Theorem 7. (Note, however, the difference in the type of the t-flags predicate.)

Let $f : \mathbb{N}^2 \to \mathbb{N}$ be such that, for each i and k,

$$f_i(k) = 2 \cdot (2^{i+1} + k - 2). \tag{58}$$

The following symbols are defined in a manner analogous to the proof of Theorem 7.

[4] In (50), we chose to use $\bigcup\{E^\infty_{i,x,0} \mid x \in X\}$. But the proof can be completed using $\bigcup\{E^\infty_{i,x,k} \mid x \in X\}$ or $\bigcup\{\bar{E}^\infty_{i,x,k} \mid x \in X\}$, for any $k < \min\{\text{num}^\infty_{i,x} \mid x \in X\}$.

– height : $\mathbb{N}^2 \to \mathbb{N}$ and $\text{height}^\infty : \mathbb{N} \to \mathbb{N}$
– num : $\mathbb{N}^2 \to \mathbb{N}$ and $\text{num}^\infty : \mathbb{N} \to \mathbb{N}$

The following symbols are defined similarly, but with f as in (58).

– $E : \mathbb{N}^3 \to \mathcal{Fin}$ and $E^\infty : \mathbb{N}^2 \to \mathcal{Fin}$
– $\bar{E} : \mathbb{N}^3 \to \mathcal{Fin}$ and $\bar{E}^\infty : \mathbb{N}^2 \to \mathcal{Fin}$

Suppose that i and s are such that $\text{height}_i^{s+1} = \text{height}_i^s + 1$. Then, by reasoning in a manner analogous to (24), it can be shown that, for each $k < \text{num}_i^{s+1}$, the following.

$$\begin{aligned} E_{i,k}^{s+1} &= E_{i,2k}^s \quad \cup \bar{E}_{i,2k}^s \quad . \\ \bar{E}_{i,k}^{s+1} &= E_{i,2k+1}^s \cup \bar{E}_{i,2k+1}^s. \end{aligned} \tag{59}$$

The partial function ψ is constructed in Figure 5. One can show Claims 8.1 through 8.6 below. The proofs are similar to those of Claims 7.1 through 7.6 (respectively).

Claim 8.1. ψ is an eps.

Claim 8.2. Suppose that i is such that $(\forall s)[i \notin R\text{-flags}^s]$. Then, for each $k < \text{num}_i^\infty$, and each p,

$$p \in (E_{i,k}^\infty \cup \bar{E}_{i,k}^\infty) \iff \psi_p = i^{(k+1)\cdot 2^h}, \tag{60}$$

where $h = \text{height}_i^\infty$.

Claim 8.3. Suppose that i is such that $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\min}$ be *least* such that $i \in R\text{-flags}^{s_{\min}+1}$. Then, there exist *distinct* ℓ and m such that (a) and (b) below.

(a) For each p,

$$p \in \bigcup\{E_{i,k}^{s_{\min}} \mid k < \text{num}_i^{s_{\min}}\} \iff \psi_p = \alpha_\ell. \tag{61}$$

(b) For each q,

$$q \in \bigcup\{\bar{E}_{i,k}^{s_{\min}} \mid k < \text{num}_i^{s_{\min}}\} \iff \psi_q = \alpha_m. \tag{62}$$

Claim 8.4. For each $p \in \text{Dst}^0$ $(= 2\mathbb{N} + 1)$ and q, if $\psi_p = \psi_q$, then $p = q$.

Claim 8.5. Suppose that i, ℓ, and s are such that $\langle i, \ell \rangle \in t\text{-flags}^s$. Then,

$$\text{rng}(\varphi_\ell) \cap E_{i,k}^s \neq \emptyset \ \wedge \ \text{rng}(\varphi_\ell) \cap \bar{E}_{i,k}^s \neq \emptyset. \tag{63}$$

Claim 8.6. Suppose that i is such that $W_i \subseteq \text{Equiv}(\psi)$. Then, $(\forall s)[i \notin R\text{-flags}^s]$.

The relation R consists initially of $\{\langle p, p \rangle \mid p \in \mathbb{N}\}$. Then, pairs are added to R as in Figure 6.

Clearly, R is a ceer. That $R \subseteq \text{Equiv}(\psi)$ follows from the $(\Rightarrow)$ directions of Claims 8.2 and 8.3.

Claim 8.7 below establishes that ψ and R satisfy (a) in the statement of the theorem, i.e., that for each translation function t into ψ, R weakly ties t into ψ. Claim 8.8 below establishes that ψ satisfies (b) in the statement of the theorem, i.e., that for each ceer R', there exists a translation function t into ψ such that R' does *not* strongly tie t into ψ.

– Stage $s = -1$. Do the following.
  - Set R-flags$^0 = \emptyset$.
  - Set t-flags$^0 = \emptyset$.
  - Set $\mathrm{Src}^0 = \mathbb{N}$.
  - Set $\mathrm{Dst}^0 = 2\mathbb{N} + 1$.
  - For each i and $k < 2^i$, set $\psi^0_{f_i(2k)} = \psi^0_{f_i(2k+1)} = i^{<k+1}$.
  - For each $p \in 2\mathbb{N} + 1$, set $\psi^0_p = \lambda x.\!\uparrow$.

– Stage $s = \langle 0, \ell\rangle$. If $\ell \in \mathrm{Src}^s$, then do the following.
  - Set $\mathrm{Src}^{s+1} = \mathrm{Src}^s \setminus \{\ell\}$.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \{\min \mathrm{Dst}^s\}$.
  - Set $\psi^{s+1}_{\min \mathrm{Dst}^s} = \alpha_\ell$.

– Stage $s = \langle i + 1, 0, -\rangle$. Determine whether there exists a k satisfying conditions (a)-(c) just below.
  (a) $i \notin R$-flags^s.
  (b) $k < \mathrm{num}^s_i$.
  (c) $W^s_i \cap (E^s_{i,k} \times \bar{E}^s_{i,k}) \neq \emptyset$.

  If such a k exists, then do the following.
  - Set R-flags$^{s+1} = R$-flags$^s \cup \{i\}$.
  - Choose any $\ell, m \in \mathrm{Src}^s$ such that $\ell \neq m$ and $i^{<2^i} \subseteq \alpha_\ell \cap \alpha_m$.
  - Let $n = \mathrm{num}^s_i$.
  - Let $\{p_0 < p_1 < \cdots < p_{n-1}\}$ be the n *least* elements of Dst^s.
  - Set $\mathrm{Src}^{s+1} = \mathrm{Src}^s \setminus \{\ell, m\}$.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \{p_0, p_1, ..., p_{n-1}\}$.
  - For each $k < n$ and $p \in E^s_{i,k}$, set $\psi^{s+1}_p = \alpha_\ell$.
  - For each $k < n$ and $q \in \bar{E}^s_{i,k}$, set $\psi^{s+1}_q = \alpha_m$.
  - For each $k < n$, set $\psi^{s+1}_{p_k} = i^{<(k+1)\cdot 2^h}$, where $h = \mathrm{height}^s_i$.

– Stage $s = \langle i + 1, \ell + 1, -\rangle$. Let $h = \mathrm{height}^s_i$. Determine whether conditions (i)-(iv) just below are satisfied.
  (i) $\ell < i$.
  (ii) $i \notin R$-flags^s.
  (iii) $\langle i, \ell\rangle \notin t$-flags^s.
  (iv) For each $k < \mathrm{num}^s_i$, $\mathrm{rng}(\varphi^s_\ell) \cap (E^s_{i,k} \cup \bar{E}^s_{i,k}) \neq \emptyset$.

  If so, then do the following.
  - Set t-flags$^{s+1} = t$-flags$^s \cup \{\langle i, \ell\rangle\}$. (Note that this implies $\mathrm{height}^{s+1}_i = \mathrm{height}^s_i + 1$.)
  - Let $n = \mathrm{num}^{s+1}_i$. (Note that, by the just previous step, $n = \mathrm{num}^s_i/2$.)
  - Let $\{q_0 < q_1 < \cdots < q_{n-1}\}$ be the n *least* elements of Dst^s.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \{q_0, q_1, ..., q_{n-1}\}$.
  - For each $k < n$ and $p \in (E^{s+1}_{i,k} \cup \bar{E}^{s+1}_{i,k})$, set $\psi^{s+1}_p = i^{(2k+2)\cdot 2^h}$.
  - For each $k < n$, set $\psi^{s+1}_{q_k} = i^{<(2k+1)\cdot 2^h}$.

Fig. 5. The construction of ψ in the proof of Theorem 8.

For each i and s, act according to the following computable condition.

– Cond. (a) $[\mathrm{height}_i^s < \mathrm{height}_i^{s+1}]$. For each $k < \mathrm{num}_i^{s+1}$ and

$$p, q \in E_{i,k}^{s+1},$$

list $\langle p, q\rangle$ into R. Similarly, for each

$$p, q \in \bar{E}_{i,k}^{s+1},$$

list $\langle p, q\rangle$ into R.

For each i, act according to the following partial computable condition.

– Cond. (b) $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\min}$ be *least* such that $i \in R\text{-flags}^{s_{\min}+1}$, and do the following. For each

$$p, q \in \bigcup\{E_{i,k}^{s_{\min}} \mid k < \mathrm{num}_i^{s_{\min}}\},$$

list $\langle p, q\rangle$ into R. Similarly, for each

$$p, q \in \bigcup\{\bar{E}_{i,k}^{s_{\min}} \mid k < \mathrm{num}_i^{s_{\min}}\},$$

list $\langle p, q\rangle$ into R.

Fig. 6. The construction of R in the proof of Theorem 8.

Claim 8.7. ψ and R satisfy (a) in the statement of the theorem, i.e., for each translation function t into ψ, R weakly ties t into ψ.

Proof of Claim. It is straightforward to verify that each $E \in \mathit{Classes}(R)$ is of one of the following three types.

– Type I. Either E is of the form
$$E_{i,k}^{\infty} \tag{64}$$
or E is of the form
$$\bar{E}_{i,k}^{\infty} \tag{65}$$
where: i is such that $(\forall s)[i \notin R\text{-flags}^s]$, and $k < \mathrm{num}_i^{\infty}$. (Intuitively, E is the result of one or more invocations of cond. (a) in Figure 6.)
– Type II. Either E is of the form
$$\bigcup\{E_{i,k}^{s_{\min}} \mid k < \mathrm{num}_i^{s_{\min}}\} \tag{66}$$
or E is of the form
$$\bigcup\{\bar{E}_{i,k}^{s_{\min}} \mid k < \mathrm{num}_i^{s_{\min}}\} \tag{67}$$
where: i is such that $(\exists s)[i \in R\text{-flags}^s]$, and $s_{\min}$ is *least* such that $i \in R\text{-flags}^{s_{\min}+1}$. (Intuitively, E is the result of zero or more invocations of cond. (a) in Figure 6, followed by a single invocation of cond. (b).)
– Type III. $E = \{p\}$, for some $p \in \mathrm{Dst}^0$ $(= 2\mathbb{N} + 1)$.

Let t be any translation function into ψ, and let ℓ be such that $\varphi_\ell = t$. Note that there are only finitely many $E \in \mathit{Classes}(R)$ of type I for which $i \leq \ell$, where i is such that $E = E_{i,k}^{\infty}$ or $E = \bar{E}_{i,k}^{\infty}$, as appropriate. Thus, to show the claim, it suffices to show that, for each $E \in \mathit{Classes}(R)$: if E is of type II or III, then $\mathrm{rng}(t) \cap E \neq \emptyset$; whereas, if E is of type I, then $\mathrm{rng}(t) \cap E \neq \emptyset$ or $i \leq \ell$ (where i is as just mentioned).

So, let $E \in \mathit{Classes}(R)$ be given. If E is of type II, then it follows from Claim 8.3($\Leftarrow$) that $\mathrm{rng}(t) \cap E \neq \emptyset$. If E is of type III, then it follows from Claim 8.4 that $\mathrm{rng}(t) \cap E \neq \emptyset$.

So, suppose that E is of type I, and that $\mathrm{rng}(t) \cap E = \emptyset$. Let i and k be such $E = E_{i,k}^{\infty}$ or $E = \bar{E}_{i,k}^{\infty}$, as appropriate. To show that $i \leq \ell$, one first assumes otherwise, by way of contradiction. One then proceeds in a manner analogous to the proof of Claim 7.7, beginning just before (42). □ (**Claim 8.7**)

Claim 8.8. ψ satisfies (b) in the statement of the theorem, i.e., for each ceer R', there exists a translation function t into ψ such that R' does *not* strongly tie t into ψ.

Proof of Claim. Suppose that ceer $R' \subseteq \mathrm{Equiv}(\psi)$. Let i be such that $W_i = R'$. Let t be any computable function such that

$$\mathrm{rng}(t) = \mathbb{N} \setminus E^{\infty}_{i,0}. \tag{68}$$

It is straightforward to show that t is a translation function into ψ. On the other hand, it is clearly the case that

$$\mathrm{rng}(t) \cap E^{\infty}_{i,0} = \emptyset. \tag{69}$$

Thus, to complete the proof, it suffices to show that, for each $E \in \mathrm{Equiv}(R)$,

$$E \cap E^{\infty}_{i,0} \neq \emptyset \;\Rightarrow\; E \subseteq E^{\infty}_{i,0}. \tag{70}$$

This can be shown in a manner analogous to the proof of Claim 7.8, beginning just after (52).

□ (**Claim 8.8**)

□ (**Theorem 8**)

Theorem 9, restated just below, is our final main result. It establishes that there can exist a single ceer that strongly ties each translation function into an eps, yet that eps's program equivalence relation can fail to be computably enumerable.

Theorem 9. There exists an eps ψ and a ceer $R \subseteq \mathrm{Equiv}(\psi)$ satisfying (a) and (b) below.

(a) For each translation function t into ψ, R strongly ties t into ψ.
(b) $\mathrm{Equiv}(\psi)$ is *not* computably enumerable.

Proof. The eps ψ is constructed below, following some necessary definitions. Let $\mathcal{Aux} \subseteq \mathcal{PartComp}$ be such that

$$\mathcal{Aux} = \mathcal{PartComp} \setminus (\{i^{<j+1} \mid i, j \in \mathbb{N}\} \cup \{\lambda x \mathbin{.} i \mid i \in \mathbb{N}\}). \tag{71}$$

Let $(\alpha_k)_{k \in \mathbb{N}}$ be a 1-1, effective numbering of $\mathcal{Aux}$.

In conjunction with ψ, the following six computable predicates are constructed.

- $\lambda i, s \mathbin{.} [i \in R\text{-flags}^s]$
- $\lambda i, j, s \mathbin{.} [\langle i, j\rangle \in t\text{-flags}^s]$
- $\lambda j, s \mathbin{.} [j \in \mathrm{Src}^s]$
- $\lambda p, s \mathbin{.} [p \in \mathrm{Dst}^s]$
- $\lambda p, i, s \mathbin{.} [p \in E^s_i]$
- $\lambda q, i, s \mathbin{.} [q \in \bar{E}^s_i]$

The purposes of these predicates are similar to those in the proofs of Theorems 7 and 8. Note, however, that in the proofs of Theorems 7 and 8, the E and $\bar{E}$ predicates were *calculated*; whereas, in this proof, they are *constructed*. The following will be clear from the construction of ψ, for each i and s.

$$E^s_i \subseteq E^{s+1}_i. \tag{72}$$
$$\bar{E}^s_i \subseteq \bar{E}^{s+1}_i. \tag{73}$$

For each i, let E^{∞}_i and $\bar{E}^{\infty}_i$ be as follows.

$$E^{\infty}_i = \bigcup\{E^s_i \mid s \in \mathbb{N}\}. \tag{74}$$

$$\bar{E}^{\infty}_i = \bigcup\{\bar{E}^s_i \mid s \in \mathbb{N}\}. \tag{75}$$

The partial function ψ is constructed in Figure 7. Claim 9.1 below establishes that ψ is an eps.

Claim 9.1. ψ is an eps.

Proof of Claim. Clearly, ψ is partial computable. Thus, it suffices to show that, for each $\zeta \in \mathcal{PartComp}$, there exists a p such that $\psi_p = \zeta$. So, let $\zeta \in \mathcal{PartComp}$ be given. Consider the following cases.

Case [$\zeta \in \mathcal{Aux}$]. Let k be such that $\alpha_k = \zeta$, and let $s = \langle 0, k\rangle$. Then, the following are easily verifiable from the construction of ψ.

– Stage $s = -1$. Do the following.
  - Set $R\text{-flags}^0 = \emptyset$.
  - Set $t\text{-flags}^0 = \emptyset$.
  - Set $\mathrm{Src}^0 = \mathbb{N}$.
  - Set $\mathrm{Dst}^0 = 3\mathbb{N} + 2$.
  - For each i, set $E_i^0 = \bar{E}_i^0 = \emptyset$.
  - For each i and j, set $\psi^0_{3\langle i,j\rangle} = i^{<2j+1}$.
  - For each i and j, set $\psi^0_{3\langle i,j\rangle+1} = i^{<2j+2}$.
  - For each $p \in 3\mathbb{N} + 2$, set $\psi^0_p = \lambda x \centerdot \uparrow$.

– Stage $s = \langle 0, k\rangle$. If $k \in \mathrm{Src}^s$, then do the following.
  - Set $\mathrm{Src}^{s+1} = \mathrm{Src}^s \setminus \{k\}$.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \{\min \mathrm{Dst}^s\}$.
  - Set $\psi^{s+1}_{\min \mathrm{Dst}^s} = \alpha_k$.

– Stage $s = \langle i + 1, 0, -\rangle$. If $i \notin R\text{-flags}^s$ and $W_i^s \cap (E_i^s \times \bar{E}_i^s) \neq \emptyset$, then do the following.
  - Set $R\text{-flags}^{s+1} = R\text{-flags}^s \cup \{i\}$.
  - Let n be *least* such that, for each $p \in (E_i^s \cup \bar{E}_i^s)$, $\psi_p \subseteq i^{<n}$.
  - Choose any $k, \ell \in \mathrm{Src}^s$ such that $k \neq \ell$ and $i^{<n} \subseteq \alpha_k \cap \alpha_\ell$.
  - Set $\mathrm{Src}^{s+1} = \mathrm{Src}^s \setminus \{k, \ell\}$.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \{\min \mathrm{Dst}^s\}$.
  - For each $p \in E_i^s$, set $\psi_p^{s+1} = \alpha_k$.
  - For each $q \in \bar{E}_i^s$, set $\psi_q^{s+1} = \alpha_\ell$.
  - Set $\psi^{s+1}_{\min \mathrm{Dst}^s} = \lambda x \centerdot i$.

– Stage $s = \langle i + 1, j + 1, -\rangle$. Determine whether conditions (i)-(iii) just below are satisfied.
  (i) $i \notin R\text{-flags}^s$.
  (ii) $\langle i, j\rangle \notin t\text{-flags}^s$.
  (iii) $\{3\langle i, j\rangle, 3\langle i, j\rangle + 1\} \subseteq \mathrm{rng}(\varphi_j^s)$.
  If so, then do the following.
  - Set $t\text{-flags}^{s+1} = t\text{-flags}^s \cup \{\langle i, j\rangle\}$.
  - Set $E_i^{s+1} = E_i^s \cup \{3\langle i, j\rangle\}$.
  - Set $\bar{E}_i^{s+1} = \bar{E}_i^s \cup \{3\langle i, j\rangle + 1\}$.
  - Let n be *least* such that, for each $p \in (E_i^{s+1} \cup \bar{E}_i^{s+1})$, $\psi_p^{s+1} \subseteq i^{<n}$.
  - For each $p \in (E_i^{s+1} \cup \bar{E}_i^{s+1})$, set $\psi_p = i^{<n}$.
  - Let $\{q_0 < q_1\}$ be the two *least* elements of Dst^s.
  - Set $\mathrm{Dst}^{s+1} = \mathrm{Dst}^s \setminus \{q_0, q_1\}$.
  - Set $\psi_{q_0}^{s+1} = i^{<2j+1}$.
  - Set $\psi_{q_1}^{s+1} = i^{<2j+2}$.

Fig. 7. The construction of ψ in the proof of Theorem 9.

– If $k \notin \mathrm{Src}^s$, then there exists a p of the form $3\langle i,j\rangle$ or $3\langle i,j\rangle+1$, for some i and j, such that $\psi_p^s = \zeta$.
– If $k \in \mathrm{Src}^s$, then $\psi^{s+1}_{\min \mathrm{Dst}^s} = \zeta$.

Case $\left[\zeta \notin \mathcal{A}ux \;\wedge\; (\exists i,j)[\zeta = i^{<2j+1}]\right]$. Let i and j be as in the case. Then, the following are easily verifiable from the construction of ψ.

– If $(\forall s)[\langle i,j\rangle \notin t\text{-flags}^s]$, then $\psi_{3\langle i,j\rangle} = \zeta$.
– If $(\exists s)[\langle i,j\rangle \in t\text{-flags}^s]$, then there exists a $p \in \mathrm{Dst}^0$ $(= 3\mathbb{N}+2)$ such that $\psi_p = \zeta$.

Case $\left[\zeta \notin \mathcal{A}ux \;\wedge\; (\exists i,j)[\zeta = i^{<2j+2}]\right]$. Similar to the previous case. □ (**Claim 9.1**)

The relation R is defined as follows.

$$R = \quad \{\langle p,p\rangle \mid p \in \mathbb{N}\} \cup \{\langle p,q\rangle \mid p,q \in E_i^\infty \;\wedge\; i \in \mathbb{N}\} \cup \{\langle p,q\rangle \mid p,q \in \bar{E}_i^\infty \;\wedge\; i \in \mathbb{N}\}. \tag{76}$$

Clearly, R is a ceer. That $R \subseteq \mathrm{Equiv}(\psi)$ follows from the $(\Rightarrow)$ directions of Claims 9.2 and 9.3.

Claim 9.6 below establishes that ψ and R satisfy (a) in the statement of the theorem, i.e., that for each translation function t into ψ, R strongly ties t into ψ. Claim 9.7 below establishes that ψ satisfies (b) in the statement of the theorem, i.e., that $\mathrm{Equiv}(\psi)$ is *not* computably enumerable.

Claim 9.2. Suppose that i is such that $(\forall s)[i \notin R\text{-flags}^s]$. Then, (a) and (b) below.

(a) Each of E_i^∞ and $\bar{E}_i^\infty$ is infinite.
(b) For each p, $p \in (E_i^\infty \cup \bar{E}_i^\infty) \;\Leftrightarrow\; \psi_p = \lambda x \centerdot i$.

Proof of Claim. Suppose that i is such that $(\forall s)[i \notin R\text{-flags}^s]$. Note that there exist infinitely many j such that $\mathrm{rng}(\varphi_j) = \mathbb{N}$. Thus, there exist infinitely many j such that $\{3\langle i,j\rangle, 3\langle i,j\rangle+1\} \subseteq \mathrm{rng}(\varphi_j^s)$, for all but finitely many s. It follows that there exist infinitely many stages of the form $\langle i+1, j+1, -\rangle$ such that all of the conditions of those stages are satisfied. Given this fact, both (a) and (b) are easily verifiable from the construction of ψ. □ (**Claim 9.2**)

Claim 9.3. Suppose that i is such that $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\min}$ be *least* such that $i \in R\text{-flags}^{s_{\min}+1}$. Then, (a) and (b) below.

(a) $E_i^\infty = E_i^{s_{\min}}$ and $\bar{E}_i^\infty = \bar{E}_i^{s_{\min}}$.
(b) There exist *distinct* k and ℓ such that (i) and (ii) below.
  (i) For each p, $p \in E_i^\infty \;\Leftrightarrow\; \psi_p = \alpha_k$.
  (ii) For each q, $q \in \bar{E}_i^\infty \;\Leftrightarrow\; \psi_q = \alpha_\ell$.

Proof of Claim. Easily verifiable from the construction of ψ. □ (**Claim 9.3**)

Claim 9.4. For each p such that

$$p \notin \bigcup\{E_i^\infty \cup \bar{E}_i^\infty \mid i \in \mathbb{N}\}, \tag{77}$$

and, for each q, if $\psi_p = \psi_q$, then $p = q$.

Proof of Claim. Easily verifiable from the construction of ψ. □ (**Claim 9.4**)

Claim 9.5. Suppose that i is such that $W_i \subseteq \mathrm{Equiv}(\psi)$. Then, $(\forall s)[i \notin R\text{-flags}^s]$.

Proof of Claim. The proof is by contrapositive. Suppose that i is such that $(\exists s)[i \in R\text{-flags}^s]$. Let $s_{\min}$ be *least* such that $i \in R\text{-flags}^{s_{\min}+1}$. Then, by the construction of ψ,

$$W_i^{s_{\min}} \cap (E_i^{s_{\min}} \times \bar{E}_i^{s_{\min}}) \neq \emptyset. \tag{78}$$

Furthermore, by Claim 9.3$(\Rightarrow)$, there exist *distinct* k and ℓ such that (a) and (b) below.

(a) For each $p \in E_i^{s_{\min}}$, $\psi_p = \alpha_k$.
(b) For each $q \in \bar{E}_i^{s_{\min}}$, $\psi_q = \alpha_\ell$.

Since α is 1-1 and $k \neq \ell$, $\alpha_k \neq \alpha_\ell$. Thus, by (78) and (a) and (b) just above, $W_i \not\subseteq \mathrm{Equiv}(\psi)$. □ (**Claim 9.5**)

Claim 9.6. ψ and R satisfy (a) in the statement of the theorem, i.e., for each translation function t into ψ, R strongly ties t into ψ.

Proof of Claim. It is straightforward to verify that each $E \in \mathit{Classes}(R)$ is of one of the following three types.

- Type I. Either E is of the form E_i^∞ or E is of the form $\bar{E}_i^\infty$ where: i is such that $(\forall s)[i \notin R\text{-flags}^s]$.
- Type II. Either E is of the form E_i^∞ or E is of the form $\bar{E}_i^\infty$ where: i is such that $(\exists s)[i \in R\text{-flags}^s]$.
- Type III. $E = \{p\}$, for some p such that

$$p \notin \bigcup\{E_i^\infty \cup \bar{E}_i^\infty \mid i \in \mathbb{N}\}. \tag{79}$$

Let t be any translation function into ψ, and let $E \in \mathit{Classes}(R)$ be given. If E is of type I, then it follows from Claim 9.2($\Leftarrow$) that $\mathrm{rng}(t) \cap E \neq \emptyset$. If E is of type II, then it follows from Claim 9.3($\Leftarrow$) that $\mathrm{rng}(t) \cap E \neq \emptyset$. If E is of type III, then it follows from Claim 9.4 that $\mathrm{rng}(t) \cap E \neq \emptyset$. □ (**Claim 9.6**)

Claim 9.7. ψ satisfies (b) in the statement of the theorem, i.e., $\mathrm{Equiv}(\psi)$ is *not* computably enumerable.

Proof of Claim. By way of contradiction, let i be such that

$$W_i = \mathrm{Equiv}(\psi). \tag{80}$$

By (80) and Claim 9.5,

$$(\forall s)[i \notin R\text{-flags}^s]. \tag{81}$$

By (81) and Claim 9.2(a), each of E_i^∞ and $\bar{E}_i^\infty$ is infinite, and, thus,

$$\text{each of } E_i^\infty \text{ and } \bar{E}_i^\infty \text{ is } \textit{non}\text{-empty}. \tag{82}$$

By (81) and Claim 9.2(b)($\Rightarrow$), for each $p \in (E_i^\infty \cup \bar{E}_i^\infty)$,

$$\psi_p = \lambda x \centerdot i. \tag{83}$$

By (80), (82), and (83),

$$W_i \cap (E_i^\infty \times \bar{E}_i^\infty) \neq \emptyset. \tag{84}$$

By (81) and (84), there exists a stage s of the form $\langle i+1, 0, -\rangle$ in which all of the conditions of that stage are satisfied. But then $i \in R\text{-flags}^{s+1}$, contradicting (81). □ (**Claim 9.7**)

□ (**Theorem 9**)

References

Ers68. Y. L. Ershov. On computable enumerations. *Algebra and Logic*, 7(5):330–346, 1968.

FKW82. R. Freivalds, E. B. Kinber, and R. Wiehagen. Inductive inference and computable one-one numberings. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 28(27-32):463–479, 1982.

Fri58. R. M. Friedberg. Three theorems on recursive enumeration. I. Decomposition. II. Maximal Set. III. Enumeration without duplication. *Journal of Symbolic Logic*, 23(3):309–316, 1958.

GG01. S. Gao and P. Gerdes. Computably enumerable equivalence relations. *Studia Logica*, 67(1):27–59, 2001.

GYY93. S. Goncharov, A. Yakhnis, and V. Yakhnis. Some effectively infinite classes of enumerations. *Annals of Pure and Applied Logic*, 60(3):207–235, 1993.

HK94. E. Herrmann and M. Kummer. Diagonals and D-maximal sets. *Journal of Symbolic Logic*, 59(1):60–72, 1994.

JST11. S. Jain, F. Stephan, and J. Teutsch. Index sets and universal numberings. *Journal of Computer and System Sciences*, 77(4):760–773, 2011.

Khu69a. A. B. Khutoretskii. On the reducibility of computable numerations. *Algebra and Logic*, 8(2):145–151, 1969.

Khu69b. A. B. Khutoretskii. Two existence theorems for computable numerations. *Algebra and Logic*, 8(4):277–282, 1969.

Kum89. M. Kummer. A note on direct sums of Friedbergnumberings. *Journal of Symbolic Logic*, 54(3):1009–1010, 1989.

Kum90. M. Kummer. An easy priority-free proof of a theorem of Friedberg. *Theoretical Computer Science*, 74(2):249–251, 1990.

Lav77. I. A. Lavrov. Computable numberings. In *Logic, Foundations of Mathematics and Computability Theory*, pages 195–206, 1977.

Mal65. A. I. Mal'cev. Positive and negative numerations. *Proceedings of the USSR Academy of Sciences*, 160(2):278–280, 1965.

Mal71. A. I. Mal'cev. Positive and negative numberings. In *The Metamathematics of Algebraic Systems*, volume 66 of *Studies in Logic and the Foundations of Mathematics*, pages 379–383. Elsevier, 1971. Translated by B. F. Wells III.

Moe12. S. E. Moelius III. Characteristics of minimal effective programming systems. In *Proceedings of Computability in Europe 2012 (CiE 2012) - How the World Computes*, Lecture Notes in Computer Science. Springer, 2012. To appear.

MWY78. M. Machtey, K. Winklmann, and P. Young. Simple Gödel numberings, isomorphisms, and programming properties. *SIAM Journal on Computing*, 7(1):39–60, 1978.

PE64. M. B. Pour-El. Gödel numberings versus Friedberg numberings. *Proceedings of the American Mathematical Society*, 15(2):252–256, 1964.

Ric81. G. A. Riccardi. The independence of control structures in abstract programming systems. *Journal of Computer and System Sciences*, 22(2):107–143, 1981.

Rog58. H. Rogers, Jr. Gödel numberings of partial recursive functions. *Journal of Symbolic Logic*, 23(3):331–341, 1958.

Rog67. H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability*. McGraw Hill, 1967. Reprinted, MIT Press, 1987.

Roy87. J. S. Royer. *A Connotational Theory of Program Structure*, volume 273 of *Lecture Notes in Computer Science*. Springer, 1987.

Sch82. B. Schinzel. On decomposition of Gödelnumberings into Friedbergnumberings. *Journal of Symbolic Logic*, 47(2):267–274, 1982.

Spr90. D. Spreen. Computable one-to-one enumerations of effective domains. *Information and Computation*, 84(1):26–46, 1990.